

Answers to Day 4 Homework

$$1. \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}[e^{t^3}]}{\frac{d}{dt}[t^2 - 1]} = \frac{3t^2 e^{t^3}}{2t} = \frac{3te^{t^3}}{2}.$$

$$2. v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{2t+5}{t^2+5t}, 6t \right\rangle \text{ so } v(2) = \left\langle \frac{9}{14}, 12 \right\rangle.$$

$$3. v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 5t^4, 12t^3 - 6t^2 \rangle.$$

$$a(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \langle 20t^3, 36t^2 - 12t \rangle, \text{ so } a(1) = \langle 20, 24 \rangle.$$

$$4. v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 3\cos\left(3t - \frac{\pi}{2}\right), 6t \right\rangle \text{ so } v\left(\frac{\pi}{2}\right) = \langle 3\cos\pi, 3\pi \rangle = \langle -3, 3\pi \rangle.$$

5. $x(t) = \int (t+1) dt = \frac{t^2}{2} + t + C.$

$x(0) = 1 = C$ so $x(t) = \frac{t^2}{2} + t + 1.$

Since $x(1) = \frac{5}{2}$ and $y(1) = \ln\left(\frac{5}{2}\right)$, Position = $\left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right).$

Or, since $x(1) = 1 + \int_0^1 (t+1) dt = \frac{5}{2}$ and $y(1) = \ln\left(\frac{5}{2}\right)$, Position = $\left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right).$

6. $x(t) = \int (1+t) dt = t + \frac{t^2}{2} + C.$ $x(0) = 5$ so $C = 5$ and $x(t) = t + \frac{t^2}{2} + 5.$

$y(t) = \int t^3 dt = \frac{t^4}{4} + D.$ $y(0) = 0$ so $D = 0$ and $y(t) = \frac{t^4}{4}.$

Position vector = $\left\langle t + \frac{t^2}{2} + 5, \frac{t^4}{4} \right\rangle.$ At $t = 2$, Position = $(9, 4).$

Or, since $x(2) = 5 + \int_0^2 (1+t) dt = 9$ and $y(2) = 0 + \int_0^2 t^3 dt = 4,$

Position = $(9, 4).$

7. When $x = 2, y = 5.$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$(2)(3) + 5 \frac{dx}{dt} = 0 \text{ so } \frac{dx}{dt} = -\frac{6}{5}$$

Or find that $\frac{dy}{dx} = -\frac{10}{x^2}.$ Then substituting into $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ gives $\frac{-10}{4} = \frac{3}{\frac{dx}{dt}}$ so that

$$\frac{dx}{dt} = -\frac{6}{5}.$$

8. $x'(t) = 3t^2 - 3t - 18 = 3(t-3)(t+2) = 0$ when $t = 3$ and $t = -2.$

$y'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0$ when $t = 3$ and $t = 1.$

The particle is at rest when $v(t) = \langle 0, 0 \rangle$ so at rest when $t = 3.$

$$\begin{aligned}
 9. \quad t^3 &= 8 & t^2 - 5t + 2 &= -4 \\
 t = 2 & & t^2 - 5t + 6 &= 0 \\
 & & (t-3)(t-2) &= 0 \\
 & & t &= 3, t = 2
 \end{aligned}$$

At $(8, -4)$ when $t = 2$

$$\left. \frac{dy}{dx} \right|_{t=2} = \left. \frac{2t-5}{3t^2} \right|_{t=2} = -\frac{1}{12}$$

Tangent line equation: $y + 4 = -\frac{1}{12}(x - 8)$

10. $5t + 3\sin t = 25$ when $t = 5.445755\dots$

$$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 5 + 3\cos t, -1 + \cos t + (8-t)\sin t \rangle$$

$$v(5.445755\dots) = \langle 7.008, -2.228 \rangle$$

11. (a) Magnitude when $t = 5$ is $\left. \sqrt{(2t)^2 + (2t^2)^2} \right|_{t=5} = \sqrt{2600}$ or $10\sqrt{26}$

(b)

$$\text{Distance} = \int_0^5 \sqrt{(2t)^2 + (2t^2)^2} dt = \int_0^5 2t\sqrt{1+t^2} dt = \frac{2}{3} (1+t^2)^{\frac{3}{2}} \Big|_0^5 = \frac{2}{3} \left(26^{\frac{3}{2}} - 1 \right)$$

(c) $\frac{dy}{dx} = \frac{2t^2}{2t} = t = \sqrt{x+3}$

12. (a) $x(t) = \int \frac{1}{t+1} dt = \ln(t+1) + C$. When $t=1, x = \ln 2$ so $C = 0$.

$$x(t) = \ln(t+1)$$

$y(t) = \int 2t dt = t^2 + D$. When $t=1, y=0$ so $D = -1$.

$$y(t) = t^2 - 1$$

$$(x, y) = (\ln(t+1), t^2 - 1)$$

(b) $t+1 = e^x$ so $t = e^x - 1$ and $y = (e^x - 1)^2 - 1 = e^{2x} - 2e^x$.

(c) Average rate of change $= \frac{y(b) - y(a)}{x(b) - x(a)} = \frac{y(4) - y(0)}{x(4) - x(0)} = \frac{15 - (-1)}{\ln 5 - \ln 1} = \frac{16}{\ln 5}$

(d) Instantaneous rate of change $= \left. \frac{dy}{dx} \right|_{t=1} = \left. \frac{2t}{\frac{1}{t+1}} \right|_{t=1} = 4$

13. (a) $\frac{dy}{dx} = \frac{2 \cos t}{3 \sin t} = \frac{2}{3} \cot t$

(b) $y - (3 + \sqrt{2}) = \frac{2}{3} \left(x - \left(2 - \frac{3\sqrt{2}}{2} \right) \right)$

(c) $x = 0$ when $t = -0.84106867\dots, 0.84106867\dots$

$$\text{length} = \int_{-0.841\dots}^{0.841\dots} \sqrt{(3 \sin t)^2 + (2 \cos t)^2} dt = 3.756 \text{ or } 3.757$$